Research Statement

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In the 2005 Topology Proceedings [1], David Bellamy posed two questions regarding the preimages of certain planar continua under specific types of maps. I began my research by learning more about this open problem. I first decided to see what his action of taking the preimage of a space "looks" like. To do this I wrote a simple program which would input a user-submitted space and output a simple drawing of a space whose image under our map would resemble the input space. The results I obtained from these simple curve examples gave me some idea of how a less basic continuum might behave under these actions.

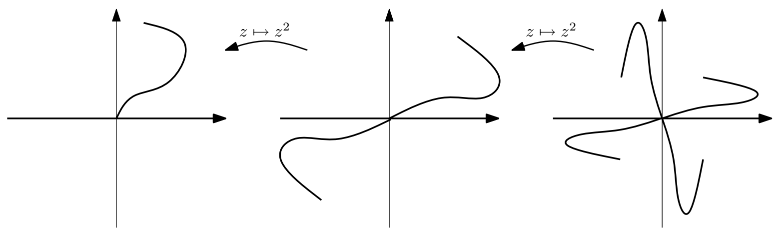

Figure 1: The preimage of a certain arc under $z \mapsto z^2$, iterated twice.

David Bellamy's questions refer to continua in the complex plane and their preimages under the map $z \mapsto z^n$. For example, the preimage of an arc under the map $z \mapsto z^2$ can be of one of three simple spaces, two of which are illustrated in Figure 1. If the arc contains the origin as an endpoint, its preimage is a single arc containing the origin, though not as an endpoint. If the arc contains the origin but not as an endpoint, its preimage is homeomorphic to the letter **X**. If the arc does not contain the origin, its preimage is the disjoint union of two arcs. While these results are somewhat obvious, they provide some insight into the preimage of certain types of continua, called *arc-like* continua. I spent some time determining which results for arcs could be generalized to arc-like continua.

Bellamy first asks whether a hereditarily indecomposable continuum which is irreducible with respect to separating the origin from ∞ must have a hereditarily indecomposable preimage. Jo Heath has produced some results [3] on specific types of 2-to-1 covering maps which she refers to as "crisp". I generalized this type of map to an n-to-1 covering map which I call "n-crisp". Using this information, I showed that if $z \mapsto z^n$ is an n-crisp map from X to Y, then X and Y are either both hereditarily indecomposable or both not hereditarily indecomposable. Furthermore, I proved that the map $z \mapsto z^n : f^{-1}(X) \to X$ is n-crisp if X separates the origin from ∞ but each proper subcontinuum of X does not. These results show that the answer to Bellamy's first question is affirmative.

Next, Bellamy asks whether the preimage of a continuum which properly contains a pseudocircle can have a hereditarily indecomposable preimage. I eventually discovered that a much easier approach to this second question was the following. Instead of starting with a continuum and looking at its preimage, we can look at a continuum as the preimage of its image. Of course, certain restrictions must be set on these "preimage continua". In particular, they must be symmetric under rotation by $\frac{2\pi}{n}$ radians. On the whole, however, this provides a much simpler approach to finding both trends and counterexamples.

For instance, to show that the preimage of an arc-like continuum can be indecomposable, we construct an indecomposable symmetric arc-like continuum. For my example, I constructed a pseudoarc M which is symmetric under rotation by π radians. Since the image under $z \mapsto z^n$ of an indecomposable continuum is indecomposable, we have reverse-engineered an effective example of our desired hypothesis.

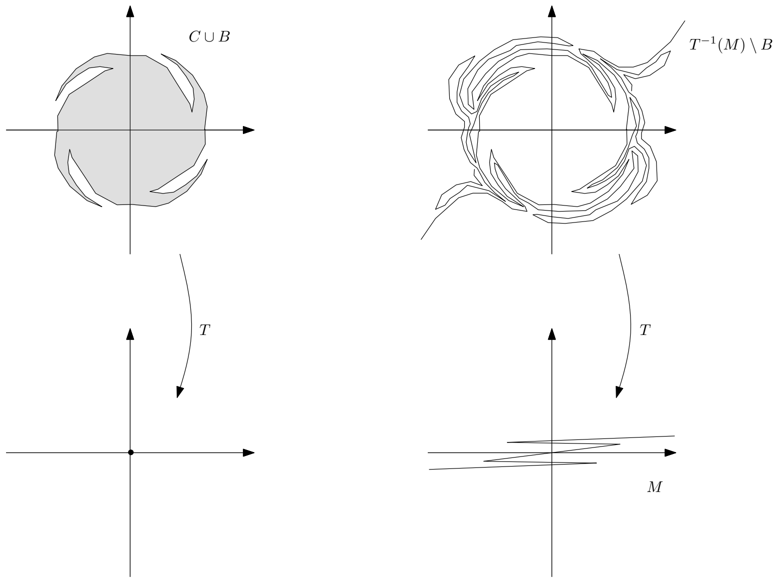

Figure 2: Using a decomposition map to construct a new continuum.

Furthermore, we can extend this example to a symmetric indecomposable continuum which properly contains a symmetric pseudocircle. RH Bing discusses [2] a "third type of hereditarily indecomposable continuum", one which properly contains a pseudocircle. We construct this continuum by first finding a hereditarily inaccessible point in a pseudoarc M, then taking a pseudocircle C and a decomposition map which takes the pseudocircle and its bounded complementary domain B to this point. The preimage of M, minus the bounded complement B of the pseudocircle, is a hereditarily indecomposable continuum (seen in Figure 2). However, this continuum is not necessarily symmetric. To solve this problem, we begin by taking the preimage under $z \mapsto z^2$) of a pseudocircle which separates the origin from ∞. As Bellamy has proven, this is a (symmetric) pseudocircle. We know there exists a decomposition map taking this pseudocircle and its bounded complementary domain to the origin, but it is not necessarily symmetric. We can, however, use this decomposition map to construct a symmetric decomposition map. Once this is constructed, we can see that the preimage of the symmetric pseudoarc under this map is again symmetric, and that it properly contains a pseudocircle (see Figure 2). Furthermore, since it is symmetric, this continuum is equal to the preimage of its image under $z \mapsto z^2$, so its image is an example of a continuum which properly contains a pseudocircle and whose preimage is hereditarily indecomposable. Then, the answer to Bellamy's second question is also affirmative.

After working on this problem, I have realized that studying the preimages of certain planar continua under various maps can lead to the discovery of very interesting types of continua. For example, what is the preimage of our symmetric pseudoarc? Is it another pseudoarc, or is it something new? I am interested in studying this material further.

My key focus in research is currently to discover new and interesting types of continua which exhibit properties that previously discovered continua do not. While I have so far restricted myself to the plane, I would like to study continua in any n-dimensional space. In my current and past research I have worked with my fellow graduate students on a regular basis, not only those in Pure Mathematics but in Applied Mathematics as well. It has been my experience that these students have just as much helpful background as any other student, and they offer a fresh perspective on the subject. In my ongoing research, I plan to incorporate other fields of mathematics into my studies.

While I am currently specializing in Continuum Theory, a special division of Point Set Topology, I am interested in broadening my studies to Algebraic Topology, Differential Topology, Geometric Topology, Complex Dynamics, and other related fields. I believe that these other areas of Topology can help me with my research.

References

[1] Bellamy, David P., "Certain Analytic Preimages of Pseudocircles", *Topology Proceedings* **Vol. 29** (2005), 19-25

[2] Bing, RH, "Concerning Hereditarily Indecomposable Continua" *Pacific J. Math* **Vol. 1 No. 1** (1951), 43-51

[3] Heath, Jo, "2-to-1 Maps with Hereditarily Indecomposable Preimages", *Proceedings of the American Mathematical Society* **Vol. 113 No. 3** (1991), 839-846